

A QUASIPERIODIC GIBBONS–HAWKING METRIC AND SPACETIME FOAM

Serdar Nergiz and Cihan Saçlıoğlu*

Physics Department, Boğaziçi University
80815 Bebek–İstanbul, Turkey

and

Physics Department, TUBITAK
Marmara Research Center
Research Institute for Basic Sciences
41470 Gebze, Turkey

Abstract

We present a quasiperiodic self-dual metric of the Gibbons–Hawking type with one gravitational instanton per spacetime cell. The solution, based on an adaptation of Weierstrassian ζ and σ functions to three dimensions, conforms to a definition of spacetime foam given by Hawking.

*Permanent address: Physics Department, Boğaziçi University, 80815 Bebek–İstanbul, Turkey

1. Introduction

Recently there has been much interest in studying integrability properties [1, 2] and underlying infinite dimensional algebras [3, 4, 5] of self-dual Einstein's equations. Earlier work on self-dual metrics, on the other hand, was focused primarily on their suspected relevance to the quantization of gravity [6, 7, 8, 9] and/or their possible effects in baryon/lepton conservation [9, 10]. In particular, Hawking [9] argued that the dominant contribution to $N(V)dV$, the number of gravitational fields with compactified spacetime volumes between V and $V + dV$, comes from metrics containing one gravitational instanton per characteristic volume, whose size is defined by a normalization constant, presumably related to Planck mass. According to Hawking, such metrics result in a foam-like structure of spacetime. The main purpose of this note is to offer an explicit example of a spacetime foam metric based on an infinite-center generalization of the Gibbons–Hawking [11] solution.

Hawking's description of spacetime foam indicates that such a metric will involve periodic or at least quasiperiodic functions of the coordinates. As Gibbons–Hawking metrics exhibit similarities to Jackiw–Nohl–Rebbi–'t Hooft [12] Yang–Mills multi-instanton solutions, periodic versions of the latter can be useful in providing insights for constructing periodic versions of the former. The first example of a periodic Yang–Mills instanton solution is due to Rossi [13], who considered an infinite string of equal size and equally spaced Jackiw–Nohl–Rebbi

instantons arranged along the Euclidean time axis. In this case, the periodic time dependence surprisingly turns out to be a gauge artifact and the solution is seen to be a static BPS monopole [14], with A_4^a playing the role of the Higgs field φ^a . The mass of the monopole is simply the action or topological charge per unit time.

Gürsey and Tze [15] took this further by writing down a self-dual Yang–Mills connection with unit instanton per spacetime cell. In the light of Rossi’s result, it is natural to interpret this solution as one representing BPS monopoles arranged on a three dimensional lattice [16]. As the lattice separation becomes smaller, such a configuration can be viewed as the much sought after monopole condensate [17], or, as the Yang–Mills counterpart of spacetime foam. Such considerations suggest that the semiclassical model for the true ground state (rather than the perturbation theory vacuum) in both General Relativity and Yang–Mills theory consists of a coherent superposition of the instantons of the theory.

One of the respects in which the two problems are dissimilar, however, is the fact that gravitational instantons, unlike Yang–Mills ones, have zero action. Thus their contribution to the path integral is not suppressed by the Boltzmann factor, hinting that the role of instantons in gravity may be even more fundamental than in Yang–Mills theory.

In mathematical terms, the Gürsey–Tze solution is based on Fueter’s quaternionic generalizations ζ^F and σ^F of the Weierstrassian quasiperiodic functions

$\zeta(z)$ and $\sigma(z)$. However, the solution can be written also in a quaternion-free form [16] which allows one to extend Weierstrassian functions to any dimension; and in particular, to the triply quasiperiodic Gibbons–Hawking $V(\vec{r})$ used here.

We review the definitions and relevant properties of $\zeta(z)$ and ζ^F in Section 2. We construct $V(\vec{r})$ and study its transformation under lattice shifts in Section 3. Topological numbers are discussed in Section 4. After some concluding observations in Section 5, we present the magnetic monopole vector potential $\vec{\omega}(\vec{r})$, in the Appendix.

2. Complex and quaternionic quasiperiodic functions:

The Gibbons–Hawking k–instanton metric has the form

$$ds^2 = \frac{1}{V}(d\tau + \vec{\omega} \cdot d\vec{r})^2 + V d\vec{r} \cdot d\vec{r} \quad (1)$$

with

$$V = \sum_{i=1}^{k+1} \frac{1}{|\vec{r} - \vec{r}_i|} \quad . \quad (2)$$

(Anti) Self-duality is imposed by choosing an $\vec{\omega}$ such that

$$\vec{\nabla} \times \vec{\omega}(\vec{r}) = \pm \vec{\nabla} V(\vec{r}) \quad . \quad (3)$$

The variable τ is restricted to $[0, 4\pi]$. Around each singularity $\vec{r} = \vec{r}_i$ of $V(\vec{r})$, one chooses a different patch and different patches are related by

$$\tau_{n+1} = \tau_n + 2\phi_n \quad , \quad (4)$$

where ϕ_n is the azimuthal angle with respect to the “z-axis” along $(\vec{r}_{n+1} - \vec{r}_n)$.

Our aim is now to construct a $V(\vec{r})$ for which the \vec{r}_i are points $\{\vec{q}\}$ belonging to a three dimensional lattice. This means the \vec{q} have the form

$$\vec{q} = n_1 \vec{q}^{(1)} + n_2 \vec{q}^{(2)} + n_3 \vec{q}^{(3)} \quad , \quad (5)$$

where the $\vec{q}^{(a)}$ ($a = 1, 2, 3$) are basis vectors of the lattice and $n_a \in \mathbf{Z}$.

One might be tempted to write simply

$$V = \sum_{n_1} \sum_{n_2} \sum_{n_3} \frac{1}{|\vec{r} - \vec{q}|} \quad , \quad (6)$$

but this expression is not convergent: An integral version of (6) exhibits a quadratic divergence of the form

$$\int_{|\vec{q}|_{min}}^{\infty} d|\vec{q}| |\vec{q}|^2 / |\vec{q}| \quad . \quad (7)$$

A similar problem in the definition of Weierstrassian elliptic functions is solved by subtraction terms. For example, in

$$\wp(z) = \frac{1}{z^2} + \sum_{\vec{n} \neq 0} \left\{ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right\} \quad , \quad (8)$$

where $\omega = n_1\omega_1 + n_2\omega_2$, in the first term one encounters a logarithmic divergence basically of the form

$$\int_{\omega_{min}}^{\infty} d|\omega| |\omega| / |\omega|^2 \quad . \quad (9)$$

Note that since the series (8) is not absolutely convergent, the parentheses are essential to its definition. Similarly, the function

$$\zeta(z) = - \int \wp(z) dz = \frac{1}{z} + \sum_{\vec{n} \neq 0} \left\{ \frac{1}{(z - \omega)} + \frac{1}{\omega} + \frac{z}{\omega^2} \right\} \quad (10)$$

necessitates two subtraction terms. Note again the importance of the brackets without which the $1/\omega$ term would sum to zero by itself, rendering the sum meaningless. The manifestly convergent form of [10] is

$$\zeta(z) = \frac{1}{z} + \sum_{\vec{n} \neq 0} \frac{z^2}{\omega^2(z - \omega)} \quad . \quad (11)$$

Finally, the σ function, whose logarithmic derivative equals $\zeta(z)$, is defined via

$$\phi \equiv \ln \sigma(z) = \ln z + \sum_{\vec{n} \neq 0} \left\{ \ln(z - \omega) - \ln(-\omega) + \frac{z}{\omega} + \frac{z^2}{2\omega^2} \right\} \quad . \quad (12)$$

We will base our generalization on the following properties of $\phi(z)$:

$$(i) \quad \partial_z \partial_{\bar{z}} \phi \propto \sum \sum \delta(x - \omega_x) \delta(y - \omega_y) \quad , \quad (13.a)$$

$$(ii) \quad \partial_z \partial_{\bar{z}} \left\{ -\ln(-\omega) + \frac{z}{\omega} + \frac{z^2}{2\omega^2} \right\} = 0 \quad , \quad (13.b)$$

(iii) three subtraction terms (up to z^2/ω^2) are needed such that for $|z| \ll |w|_{min}$,

$$\phi(z) \cong \ln z + \sum \sum O\left(\frac{z^3}{\omega^3}\right) + O\left(\frac{z^4}{\omega^4}\right) + \dots \quad . \quad (13.c)$$

The last property is dictated by convergence requirements and simple dimensional analysis: the terms in the sum cannot involve powers of ω higher than ω^{-3} ; the dimensionlessness of $\phi(z)$ then implies the form (13.c).

Fueter has constructed analogues of (11) and (12) by using a quaternionic variable $\mathbf{x} \equiv Ix_0 - i\vec{\sigma} \cdot \vec{x}$ instead of z . However, a suitable adaptation of (13) allows the definition of quasiperiodic functions over any \mathbf{R}^n , suggesting quaternionic techniques are not essential to the construction in \mathbf{R}^4 . Indeed, using (13) with $q_\mu = n_0 q_\mu^{(0)} + n_1 q_\mu^{(1)} + n_2 q_\mu^{(2)} + n_3 q_\mu^{(3)}$, one finds [10],

$$\rho(x) = \frac{1}{x^2} + \sum_{\{q\} \neq 0} \left\{ \frac{1}{(x-q)^2} - \frac{1}{q^2} - \frac{2q \cdot x}{q^4} - \frac{1}{q^6} (4(q \cdot x)^2 - q^2 x^2) \right\} \quad (14)$$

for the four-dimensional counterpart of $\phi(z)$. The analogue of (11) is the vector field defined by

$$\zeta_\mu^F(x) = \partial_\mu \rho(x) \quad . \quad (15)$$

In Fueter's quaternionic notation this is converted to the quaternion-valued function

$$\zeta^F(x) = I \partial_o \rho + i \vec{\sigma} \cdot \vec{\nabla} \rho \quad (16)$$

which restores the formal similarity to (11).

3. The triply quasiperiodic Gibbons–Hawking potential:

It is now a straightforward matter to apply (13.a,b,c) in \mathbf{R}^3 to obtain

$$V(\vec{r}) = \frac{1}{r} + \sum_{\{\vec{q}\} \neq 0} \sum_{\{\vec{q}\} \neq 0} \left\{ \frac{1}{|\vec{r} - \vec{q}|} - \frac{1}{|\vec{q}|} \left[1 + \frac{\vec{q} \cdot \vec{r}}{q^2} + \frac{1}{2q^4} (3(\vec{q} \cdot \vec{r})^2 - q^2 r^2) \right] \right\} . \quad (17)$$

Note again that (14) and (17) are only meaningful if the outer parentheses are respected. The last two terms in both (14) and (17) cannot be separately summed to zero anymore than the $1/\omega$ in (10) can.

The analogue of $\zeta(z)$ is now the vector field $\vec{\nabla}V(\vec{r})$. It is well known that under lattice shifts, ζ obeys the quasiperiodic transformation law

$$\zeta(z + \omega_{1,2}) = \zeta(z) + \eta_{1,2} , \quad (18)$$

where $\eta_{1,2}$ are constant complex numbers obeying Legendre's relation

$$\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i . \quad (19)$$

Integrating (18) and using the oddness of $\sigma(z)$ one has

$$\sigma(z + \omega_{1,2}) = -\sigma(z) \exp\left[\eta_{1,2}\left(z + \frac{1}{2}\omega_{1,2}\right)\right] . \quad (20)$$

To derive \mathbf{R}^3 generalizations of (18)-(20) we note $\nabla^2 V$ is a perfectly triply periodic arrangement of δ - functions; thus we may integrate it once to obtain

$$\vec{\nabla}V(\vec{r} + \vec{q}^{(a)}) = \vec{\nabla}V(\vec{r}) + \vec{\eta}^{(a)} , \quad (a = 1, 2, 3) , \quad (21)$$

where the $\vec{\eta}^{(a)}$ are constant vectors. Integrating once more and using the fact that $V(-\vec{q}^{(a)}/2) = V(\vec{q}^{(a)}/2)$, which incidentally means $\eta_i^{(a)} = 2(\partial_i V)_{\vec{r}=\vec{q}^{(a)}/2}$, we find

$$V(\vec{r} + \vec{q}^{(a)}) = V(\vec{r}) + \vec{\eta}^{(a)} \cdot (\vec{r} + \frac{\vec{q}^{(a)}}{2}) \quad (\text{no sum over } a) \quad . \quad (22)$$

This is clearly the counterpart of (20). Integrating $\vec{\nabla}V$ over the surface (∂cell) of a period cell yields

$$\sum_{a=1}^3 \sum_{b=1}^3 \sum_{c=1}^3 \frac{1}{2} \epsilon^{abc} \vec{\eta}^{(a)} \cdot (\vec{q}^{(b)} \times \vec{q}^{(c)}) = -4\pi \quad . \quad (23)$$

This replaces Legendre's relation (19) in three dimensions. We may also iterate (22) to find

$$V(\vec{r} + m\vec{q}^{(a)}) = V(\vec{r}) + \vec{\eta}^{(a)} \cdot (m\vec{r} + \frac{m^2}{2}\vec{q}^{(a)}) \quad . \quad (24)$$

4. Topological numbers:

As we are dealing with self-dual curvature two-forms R^a_b , the Euler class

$$e = \frac{1}{32\pi^2} \epsilon_{abcd} R^a_k \wedge R^c_l \delta^{bk} \delta^{ld} \quad (25)$$

and the Pontrjagin class

$$p_1 = -\frac{1}{8\pi^2} R^a_b \wedge R^b_a \quad (26)$$

are simply proportional to each other. The topological numbers related to (25) and (26) are the Euler characteristic χ and the signature τ , respectively. For a $(k+1)$ -center Gibbons–Hawking metric one has [19, 20]

$$\tau_k = \frac{1}{3} \int p_1 - \frac{1}{k+1} \sum_{n=1}^k \cot^2\left(\frac{n\pi}{k+1}\right) = -k \quad (27)$$

and

$$\chi = k + 1 \quad . \quad (28)$$

Remarkably, the analogy between Jackiw–Nohl–Rebbi–t Hooft and Gibbons–Hawking instantons carries over to the expressions for the topological charges. In the Yang–Mills case, the connection

$$A_\mu = i\vec{\sigma}_{\mu\nu}\partial_\nu \ln \rho \quad (29)$$

with

$$\rho = \sum_{i=1}^{n+1} \frac{\lambda_i^2}{(x - x_i)^2} \quad (30)$$

gives rise to a self-dual field strength, which results in the topological charge density

$$-\frac{d^4x}{16\pi^2} \text{Tr}(\tilde{F}_{\mu\nu}F_{\mu\nu}) = -\frac{d^4x}{16\pi^2} \square\square \ln \rho \quad . \quad (31)$$

The integral of this expression gives n when converted into a surface integral over a large S^3 containing all the singularities plus infinitesimal S^3 's around each singularity. In the Gibbons–Hawking parametrization, a lengthy but straightforward calculation yields

$$p_1 = -\frac{1}{16\pi^2} d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3 \triangle\triangle \frac{1}{V} \quad . \quad (32)$$

Just as in the Yang–Mills computation based on (31), one can convert the volume integral of (32) to the surface integral

$$\frac{1}{3} \int p_1 = -\frac{1}{48\pi^2} \int d\tau \int d\vec{\sigma} \cdot \vec{\nabla} \left(\frac{2\vec{\nabla}V \cdot \vec{\nabla}V}{V^3} \right) = -\frac{1}{6\pi^2} \int d\vec{\sigma} \cdot \vec{\nabla} \left(\frac{\vec{\nabla}V \cdot \vec{\nabla}V}{V^3} \right) \quad , \quad (33)$$

where the surface consists of a large S^2 containing all the singularities of V plus $k + 1$ infinitesimal S^2 's around each singularity. The result is

$$\begin{aligned}\tau_k &= \frac{2}{3}\left(\frac{1}{k+1} - (k+1)\right) - \frac{1}{k+1} \sum_{n=1}^k \cot^2\left(\frac{n\pi}{k+1}\right) \\ &= \frac{2}{3}\left(\frac{1}{k+1} - (k+1)\right) - \frac{k(k-1)}{3(k+1)} = -k\end{aligned}\tag{34}$$

as expected. For the solution corresponding to (17) with a singularity per unit cell, $\tau(\text{per cell}) = -1$. Although we have been working with a lattice in \mathbf{R}^3 , the fourth variable is already periodic with a period 4π ; hence this is to be regarded as topological number per spacetime cell. Thus we have a close analogue of the self-dual Yang–Mills solution due to Gürsey and Tze [15].

5. Concluding remarks:

The quasiperiodic functional form given in (17) defines a different metric for each choice of three dimensional lattice. Furthermore, just as one can superpose k terms of the form $|\vec{r} - \vec{r}_i|^{-1}$, one can superpose V 's for different lattices, or for isometric lattices with periods that are multiples of some irreducible basis vectors. Solutions such as (17) can then be used as a building blocks for more complex types of space-time foam. On the other hand, there must exist another class of manifolds with similar periodic properties, but which cannot be obtained from (17). These correspond to asymptotically locally Euclidean self-dual manifolds whose boundaries are other spherical forms of S^3 related to the dihedral groups D_k of order k and to certain discrete groups. Since these metrics are not known, the treatment in this paper cannot at present be extended to the corresponding instantons. We will nevertheless venture some speculations concerning the lattices that are likely to be encountered.

It is natural to expect that lattices that correspond to the tightest packing of spheres might play a special role. For example, in the Copenhagen model [21] for the Yang–Mills vacuum one first considers two dimensional lattices of chromomagnetic vortex tubes, which yield a vacuum energy below that of the perturbation theory vacuum when one loop corrections are taken into account. The energy is then lowered further when the tightest packing corresponding to the hexagonal root lattice of $SU(3)$ is chosen.

In our four dimensional problem the tightest packing lattice is the root lattice of $SO(8)$. However, the Dynkin diagram of $SO(8)$ makes it clear that this is not a possible choice for our Gibbons–Hawking class of metrics based on the cyclic groups A_k . The reason for this is obvious: the lattice vector in the τ direction is orthogonal to all the $\vec{q}^{(a)}$ while none of the $SO(8)$ simple roots has this property. The tightest packing available for the A_k class considered here obtains when the $\vec{q}^{(a)}$ are taken as the simple roots of $SU(4)$. We conjecture that the $SO(8)$ lattice may be relevant for metrics based on the D_k family. If true, this would imply that the D_k metrics (unlike A_k ones) cannot be parametrized in terms of functions independent of τ such as $V(\vec{r})$ and $\vec{\omega}(\vec{r})$. Pursuing the analogy between Jackiw–Nohl–Rebbi–’t Hooft and Gibbons–Hawking instantons, it is tempting to regard the D_k metrics as analogues of the ADHM [22] instanton solution for which the group space orientations of the instantons are in general not parallel.

Appendix:

We work in a gauge where the monopole centered at the point $\vec{q} = n_1\vec{q}^{(1)} + n_2\vec{q}^{(2)} + n_3\vec{q}^{(3)}$ has a string originating from \vec{q} and lying parallel to the negative z -axis. In order to avoid singularities, the z -axis must not be aligned with any of the \vec{q} ; the reason for this restriction will be apparent from the expression below. The construction of the vector potential $\vec{\omega}$ then proceeds along the lines leading

to (17). Thus one starts with

$$\hat{z} \times \left\{ \frac{\vec{r}}{r} \frac{1}{z+r} + \sum_{\{\vec{q} \neq 0\}} \sum_{\{\vec{q} \neq 0\}} \frac{\vec{r} - \vec{q}}{|\vec{r} - \vec{q}|} \frac{1}{(z - q_z) + |\vec{r} - \vec{q}|} \right\} , \quad (A.1)$$

which represents monopole vector potentials centered at the origin and at the points $\{\vec{q}\}$. One then Taylor expands the terms in the sum for small (x, y, z) up to and including terms of quadratic order. The terms so obtained are then subtracted from (A.1), yielding the expression

$$\vec{\omega} = \hat{k} \times \left\{ \frac{\vec{r}}{r} \frac{1}{z+r} + \sum_{\{\vec{q} \neq 0\}} \sum_{\{\vec{q} \neq 0\}} \left[\frac{\vec{r} - \vec{q}}{|\vec{r} - \vec{q}|} \frac{1}{(z - q_z) + |\vec{r} - \vec{q}|} - \vec{\Omega}_q \right] \right\} , \quad (A.2)$$

where

$$\begin{aligned} \vec{\Omega}_q = & \frac{1}{q(q - q_z)} \left\{ -\vec{q} + \left(\frac{1}{q^2} [\vec{r} q^2 - \vec{q}(\vec{q} \cdot \vec{r})] - \frac{\vec{q}[(\vec{q} \cdot \vec{r}) - zq]}{q(q - q_z)} \right) \right. \\ & + \frac{1}{2} \left[\vec{q} \frac{r^2}{q^2} + 2\vec{r} \frac{(\vec{q} \cdot \vec{r})}{q^2} - 3\vec{q} \frac{(\vec{q} \cdot \vec{r})^2}{q^4} + 2[\vec{r} q^2 - \vec{q}(\vec{q} \cdot \vec{r})] \frac{[(\vec{q} \cdot \vec{r}) - zq]}{q^3(q - q_z)} \right. \\ & - \vec{q} \left(-\frac{[\hat{k} \cdot (\vec{r} \times \vec{q})]^2}{q^3(q - q_z)} + \frac{2(\vec{q} \cdot \vec{r})^2}{q^2(q - q_z)^2} - \frac{2z(\vec{q} \cdot \vec{r})}{q^3(q - q_z)^2} (q_z^2 + 2q^2 - qq_z) \right. \\ & \left. \left. \left. - \frac{r^2 q_z^2}{q^3(q - q_z)} + \frac{z^2(q - q_z)}{q^3} + \frac{z^2 q_z}{q^3(q - q_z)^2} (q_z^2 + 4q^2 - 3qq_z) \right) \right] \right\} . \end{aligned} \quad (A.3)$$

It is not difficult to verify that this $\vec{\omega}$ satisfies

$$\vec{\nabla} \times \vec{\omega} = \vec{\nabla} V \quad (A.4)$$

for the V given in (17).

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